Computer Science Department

TECHNICAL REPORT

"Decision Algorithms for Some Fragments
of Analysis and Related Areas"

by
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Technical Report #217
May, 1986

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[†] This research is supported in part by the National Science Foundation under Grant No. MCS-8300631.

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1. Introduction

This paper gives (relatively straightforward) decision algorithms for some fragmentary sublanguages of elementary analysis and related fields. It is felt that the algorithms presented could form a useful part of the 'initial endowment' of a computerized proof verifier intended to facilitate proofs in real analysis. The specific languages treated are

- (i) The (unquantified) 'extended Tarski' theory of reals with additional variables designating continuous real-valued functions. Here variables of two types, respectively representing real numbers and real-valued functions, exist, along with operation signs designating real addition, subtraction, multiplication and division, and comparison. Moreover, the theory also allows addition and subtraction of functions, as well as their evaluation at specified real numbers, and moreover allows positivity, monotonicity and convexity of functions to be asserted.
- (ii) A multi-dimensional extension of theory (i).
- (iii) More abstract theories of totally and partially ordered sets and monotone functions on them.

In all cases, decidability is shown using a standard reduction process, which proceeds by applying a series of transformations to an

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initial formula in such a way as to preserve satisfiability but which is such that the resulting formula belongs to a theory for which a decidability algorithm is already known. For example, for language (i) we reduce formulae to formulae in the elementary theory of reals, which is known to be decidable.

2. The normalization process

Let T be an unquantified theory with equality $=$, variables $x, y, \dots$, function symbols $f_1, f_2, \dots$, and predicate symbols $P_1, P_2, \dots$.

Definition 2.1. A formula ϕ of T is in normal form if:

- (i) all terms occurring in ϕ have the simple form $f(x_1, \dots, x_n)$ or just x , where $x, x_1, \dots, x_n$ are variables and f is a function symbol;
- (ii) all atoms are either of the form $x = t$, where x is a variable and t a term, or of the form $P(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are variables of appropriate types and P is a predicate symbol which applies to variables of these types.

In dealing with the decidability problems of any theory, it is advantageous to be able to concentrate on formulae in normal form. To transform any formula into a normal form formula such that the initial and resulting formula are equisatisfiable, we use the following easy result.

Lemma 2.2. There is an effective procedure to transform any formula ϕ in T to a formula ψ in normal form such that ϕ and ψ are equisatisfiable.

Proof: Consider the following

Normalization Algorithm

WHILE there is an atom of the form $t_s = t$, or there is an atom of the form $P_1(t_1, \dots, t_{s-1}, t_s, t_{s+1}, \dots, t_n)$ where $t_1, \dots, t_n, t$ are terms and t_s, t are not variables DO

Let x_{t_s} be a new variable of the same type as t_s and not occurring in ϕ . Add to ϕ the formula $x_{t_s} = t_s$ and substitute the variable x_{t_s} wherever the term t_s occurs in ϕ .

END WHILE;

[Comment: At this point part (ii) of Definition 2.1 holds. To satisfy also part (i) we perform the following loop.]

WHILE there is a proper subterm t_s which is not a variable occurring in a term $f(t_1, \dots, t_{s-1}, t_s, t_{s+1}, \dots, t_n)$ DO:

Let x_{t_s} be a variable of the same type as the term t_s and not occurring in ϕ .

Add to ϕ the formula $x_{t_s} = t_s$ and substitute the variable x_{t_s} wherever a term t_s occurs.

END WHILE;

It is clear that each step of the above algorithm preserves satisfiability. Therefore the final formula ψ produced by the algorithm satisfies the assertion of the lemma.

Definition 2.3. A normal form formula ϕ of τ is a standard normal form formula if it is a conjunction of literals of the types

(=) $x_1 = x_2, \quad x_1 = f(y_1, \dots, y_n)$

($\neq$) $x_1 \neq x_2$

(P) $P(x_1, \dots, x_n)$

($\neg$ P) $\neg P(x_1, \dots, x_n),$

where $x_1, \dots, x_n, y_1, \dots, y_n$ are variables.

1. The first part of the report

is devoted to a general

description of the

method used in the investigation

and the results obtained

in the course of the work

and the conclusions drawn from the

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Let S be the class of all formulae of T in standard normal form. Then we have the following

Lemma 2.4. T is decidable if and only if S is decidable.

Proof: Clearly any algorithm for testing satisfiability of formulae in T is an algorithm for testing satisfiability of formulae in S . Conversely let A be an algorithm which tests satisfiability in S , and let ϕ be any formula in T . By applying the normalization process to ϕ , we get a normal form formula ψ such that ϕ and ψ are equisatisfiable. At this point we can put ψ in disjunctive normal form obtaining a formula $\psi_1 \vee \dots \vee \psi_n$, where all ψ_i are conjunctions, and we may assume that each ψ_i is in standard normal form, because any literals in it of the type $\neg (x_1 = f(y_1, \dots, y_n))$ can be replaced by the conjunction $\neg (x_1 = x_2) \ \& \ (x_2 = f(y_1, \dots, y_n))$ by introducing a variable x_2 which occurs nowhere else. Since

$$\begin{aligned} \phi \text{ is satisfiable} &\leftrightarrow \psi \text{ is satisfiable} \\ &\leftrightarrow \psi_i \text{ is satisfiable for some } i, \end{aligned}$$

and since all transformations used to build conjunctions $\psi_1, \dots, \psi_n$ are effective, our lemma follows. Q.E.D.

In the following sections we will use these elementary concepts and results freely, without specific reference.

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3. 'Extended Tarski' theory of reals with continuous functions.

In the extension of Tarski theory of reals considered in this section we allow two kinds of variables, namely real variables, $x, y, \dots$, and function variables $f, g, \dots$. Constants 0, 1, both numerical and functional are also allowed. We distinguish two kinds of terms, namely numerical terms and functional terms, as follows:

Definition 3.1. Numerical terms are defined recursively by:

- (i) Any real variable $x, y, \dots$ or constant 0, 1 is a numerical term.
- (ii) If t_1, t_2 are numerical terms, then $(t_1 + t_2)$, $(t_1 - t_2)$, $(t_1 \cdot t_2)$, (t_1 / t_2) are numerical terms.
- (iii) If t is a numerical term and f is a function variable, then $f(t)$ is a numerical term.
- (iv) An expression is a numerical term just if it can be shown to be so on the basis of (i), (ii), (iii) above.

Functional terms are defined recursively as follows:

- (i') Any function variable $f, g, \dots$ and constants 0, 1 are functional terms.
- (ii') If t_1, t_2 are functional terms, then $(t_1 + t_2)$, $(t_1 - t_2)$ are functional terms.
- (iii') An expression is a functional term just if this follows from (i') and (ii').

Having defined the terms allowed by our small language, we now proceed to define the family of formulae comprising the extension of Tarski's theory of reals that we want to discuss. First of all we define 'atoms'. An atom is an expression of one of the following forms:

$t_1 = t_2$, $t_1 < t_2$, $\eta_1 = \eta_2$, $\eta_1 < \eta_2$, $\text{up}(\eta_1, t_1, t_2)$,
 $\text{strict_up}(\eta_1, t_1, t_2)$, $\text{down}(\eta_1, t_1, t_2)$, $\text{strict_down}(\eta_1, t_1, t_2)$,

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$\text{convex}(\eta_1, t_1, t_2)$, $\text{concave}(\eta_1, t_1, t_2)$, where t_1, t_2 are numerical terms and η_1, η_2 are functional terms. (The intended meaning of up , strict_up , etc. will be defined below.) Finally, we define a formula in the theory of Reals with Monotone and Convex Functions (RMCF) to be any boolean combination of atoms.

Let ϕ be a formula in RMCF.

Definition 3.2. An interpretation M of the symbols of ϕ is a real model of ϕ if:

- (1) For any real variable x , the value Mx associated to x is a real number.
- (2) For any function variable f , Mf is an everywhere defined continuous real function of a single real variable.
- (3) The truth value assigned to ϕ by the following rule is the value true:

Let t_1, t_2 be numerical terms, let η_1, η_2 be functional terms, and assume that $Mt_1, Mt_2, M\eta_1, M\eta_2$ are interpretations assigned to them by the model M in the standard way. Then

- (3a) $t_1 = t_2$ (resp. $t_1 < t_2$) is true if and only if
 $Mt_1 = Mt_2$ (resp. $Mt_1 < Mt_2$)
- (3b) $\eta_1 = \eta_2$ (resp. $\eta_1 < \eta_2$) is true if and only if
 $M\eta_1 = M\eta_2$ (resp. $M\eta_1 < M\eta_2$) pointwise
- (3c) $\text{up}(\eta_1, t_1, t_2)$ (resp. $\text{strict_up}(\eta_1, t_1, t_2)$) is true
if and only if $Mt_1 \geq Mt_2$, or $Mt_1 < Mt_2$ and the function
 $M\eta_1$ is monotone nondecreasing (resp. strictly increasing) in
the interval $[Mt_1, Mt_2]$.
- (3d) $\text{convex}(\eta_1, t_1, t_2)$ is true if and only if $Mt_1 \geq Mt_2$ or

$Mt_1 < Mt_2$ and the function Mn_1 is convex in the interval $[Mt_1, Mt_2]$.

- (3e) The truth values of $\text{down}(\eta_1, t_1, t_2)$, $\text{strict_down}(\eta_1, t_1, t_2)$, $\text{concave}(\eta_1, t_1, t_2)$ are defined in a manner completely analogous to the corresponding definitions (3c), (3d), with the sole difference that increasing and decreasing have to be interchanged and concave has to be read in place of convex.

Observe that formulae $t_1 = t_2 - t_3$, $t_1 > t_2$ where t_1, t_2, t_3 are either numerical or functional terms are respectively equisatisfiable with $t_2 = t_1 + t_3$ and $t_1 = t_2 + v$ & $v > 0$, where v is a variable of the appropriate type. Likewise formula $t_1 = t_2 / t_3$ is equisatisfiable with $t_3 \neq 0$ & $t_2 = t_1 \cdot t_3$. Moreover if x and y are real variables, then $x \neq y$ is equivalent to $x < y \vee x > y$. Therefore it is easy to see that by applying an elementary normalization process of the kind described in the preceding section we can without loss of generality consider only sets of clauses each of which has one of the following types:

(+) $x = y + z$

(.) $x = y \cdot z$

(>) $x > 0$

(f) $x = f(y)$

(f+) $f = g + h$

(f $\neq$) $f \neq g$

(f>) $f > 0$

(u-d) $\text{up}(f, x, y), \text{down}(f, x, y)$

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$(\text{strict}) \quad \text{strict_up}(f,x,y), \quad \text{strict_down}(f,x,y)$
 $(\neg \mid \text{u-d}) \quad \neg \mid \text{up}(f,x,y), \quad \neg \mid \text{down}(f,x,y)$
 $(\neg \mid \text{strict}) \quad \neg \mid \text{strict_up}(f,x,y), \quad \neg \mid \text{strict_down}(f,x,y)$
 $(\text{con}) \quad \text{convex}(f,x,y), \quad \text{concave}(f,x,y)$
 $(\neg \mid \text{con}) \quad \neg \mid \text{convex}(f,x,y), \quad \neg \mid \text{concave}(f,x,y),$

where x,y,z are real variables and f,g,h are function variables,

In order to show that the theory RMCF is decidable, we will exhibit a simple reduction that can be applied to a given initial formula, and that yields a formula with the following properties:

- (a) no function variable occurs in the resulting formula (therefore in particular only clauses of the types $(+, \cdot, >)$ can occur);
- (b) the initial and the resulting formulae are equisatisfiable.

Note that (a) and (b) combined with the basic result that formulae involving only clause of the types $(+, \cdot, >)$ are decidable (see [Tar51], [Col75]) implies the decidability of the theory RMCF.

To do this, let ϕ be a formula of RMCF, which by Lemma 2.4 and by the foregoing discussion can be assumed to be a conjunction of clauses of the types $(+)-(\neg \mid \text{con})$ listed above. We begin by treating all clauses containing a negation. The reason for this is that any negative clause involving a function symbol (except as a numerical term; see Definition 3.1) can be viewed as an implicit existential assertion whose quantifier is easy to eliminate by adding all instantiation of the quantified body to our set of clauses. More specifically, we proceed as follows:

Simplification Step 1:

- (1a) For any literal $f \neq g$ occurring in ϕ , introduce three new

variables x_1, x_2, y and replace $f \neq g$ by the formula $x_1 = f(y) \ \& \ x_2 = g(y) \ \& \ x_1 \neq x_2$ (as previously noted, we can use $x_1 > x_2 \vee x_2 > x_1$ in place of $x_1 \neq x_2$ but for reasons of clarity we do not insist on noting such simplifications at every step of what follows).

- (1b) Replace any literal $\neg \mid \text{up}(f, x, y)$ (resp. $\neg \mid \text{strict_up}(f, x, y)$) in ϕ by a formula

$$A \ \& \ w_1 > w_2 \quad (\text{resp. } A \ \& \ w_1 \geq w_2),$$

where $A \equiv x \leq z_1 < z_2 \leq y \ \& \ w_1 = f(z_1) \ \& \ w_2 = f(z_2)$ and where z_1, z_2, w_1, w_2 are newly introduced variables. Make analogous substitutions for literals $\neg \mid \text{down}(f, x, y)$ and $\neg \mid \text{strict_down}(f, x, y)$.

- (1c) Replace any literal $\neg \mid \text{convex}(f, x, y)$ (resp. $\neg \mid \text{concave}(f, x, y)$) by the formula

$$B \ \& \ (w_2 - w_1)(z_3 - z_1) > (z_2 - z_1)(w_3 - w_1)$$

$$(\text{resp. } B \ \& \ (w_2 - w_1)(z_3 - z_1) < (z_2 - z_1)(w_3 - w_1)),$$

where

$$B \equiv x \leq z_1 < z_2 < z_3 \leq y \ \& \ w_1 = f(z_1) \ \& \ w_2 = f(z_2) \ \& \ w_3 = f(z_3)$$

and where $z_1, z_2, z_3, w_1, w_2, w_3$ are newly introduced real variables. (The formula above just expresses the existence of three points z_1, z_2, z_3 , such that $(z_2, f(z_2))$ lies above (resp. below) the straight line joining the two points $(z_1, f(z_1))$ and $(z_3, f(z_3))$.)

1. The first part of the document
describes the general situation
of the country and the
state of the economy.
It also mentions the
political situation and the
state of the army.
The second part of the document
describes the situation in the
provinces and the state of the
economy in each province.
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The third part of the document
describes the situation in the
cities and the state of the
economy in each city.
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state of the army in each city.
The fourth part of the document
describes the situation in the
villages and the state of the
economy in each village.
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situation in each village and the
state of the army in each village.

Let ϕ_1 be the resulting formula; then it is easy to see that ϕ and ϕ_1 are equisatisfiable. Indeed clearly any model of ϕ is a model of ϕ_1 . Conversely let M be a model of ϕ_1 . Then it follows easily, e.g., that if $f \neq g$ is in ϕ then $Mf \neq Mg$. In fact by step (1a) a formula $x_1 = f(y) \ \& \ x_2 = g(y) \ \& \ x_1 \neq x_2$ is present in ϕ_1 . Therefore since M is a model for ϕ we have $Mx_1 = (Mf)(My)$, $Mx_2 = (Mg)(My)$ and $Mx_1 \neq Mx_2$, that is $(Mf)(My) \neq (Mg)(My)$, which in particular shows that $(Mf) \neq (Mg)$. It can be established in much the same way that steps (1b) and (1c) do not disrupt satisfiability of ϕ . Normalizing ϕ_1 again and using Lemma 2.4, we can assume without loss of generality that ϕ_1 is a conjunction of literals of the following types $(+, \cdot, >, f, f+, f>, u-d, \text{strict}, \text{con})$.

We now show how all literals containing function variables can be eliminated. For this, we apply the following

Preparatory Step 2:

For any real variable x which occurs as argument of any function variable or as a bound in a literal of the type $(u-d, \text{strict}, \text{con})$ and for any function variable f occurring in ϕ_1 , introduce a new variable $z_{x,f}$ and add the formula $z_{x,f} = f(x)$ to ϕ_1 , unless a literal $y = f(x)$ exists already.

Let ϕ_2 be the resulting formula. Since each newly introduced variable $z_{x,f}$ appears in just one equality literal, it is clear that ϕ_1 and ϕ_2 are still equisatisfiable at this stage. Note that Step 2 is applied simply in preparation for the elimination of all $(u-d, \text{strict}, \text{con})$ clauses; it serves only to ensure that all potentially significant implicit function evaluations appear explicitly. We are now ready for step 3.

 

$$f(x) = \int_0^x \frac{1}{1+t^2} dt$$

$$f(x) = \arctan x$$

 

Simplification Step 3

- (3a) For each quadruple of literals $x = f(y)$, $x' = g(y)$, $x'' = h(y)$, $f = g + h$, add $x = x' + x''$.
- (3b) For each pair of literals $x = f(y)$, $f > 0$ add $x > 0$.
- (3c) For each pair of literals $x = f(y_1)$, $x' = f(y_2)$ add the formula $y_1 = y_2 \rightarrow x = x'$.
- (3d) For each triple of literals $x = f(y)$, $x' = f(y')$, $\text{up}(f,v,w)$ (resp. $\text{down}(f,v,w)$) add the formula $v \leq y < y' \leq w \rightarrow x \leq x'$ (resp. $v \leq y < y' \leq w \rightarrow x \geq x'$)
- (3e) For each triple of literals $x = f(y)$, $x' = f(y')$, and $\text{strict_up}(f,v,w)$ (resp. $\text{strict_down}(f,v,w)$) add the formula $v \leq y < y' \leq w \rightarrow x < x'$ (resp. $v \leq y < y' \leq w \rightarrow x > x'$)
- (3f) For each quadruple of literals $x = f(y)$, $x' = f(y)$, $x'' = f(y'')$, $\text{convex}(f,v,w)$ (resp. $\text{concave}(f,v,w)$) add the formula:
$$v \leq y' < y < y'' \leq w \rightarrow (y-y') \cdot (x''-x') \geq (x-x') \cdot (y''-y')$$

(resp. $v \leq y' < y < y'' \leq w \rightarrow (y-y') \cdot (x''-x') \leq (x-x') \cdot (y''-y')$).
(Observe that the implication above says just that the 'point' (y,x) is above (resp. under) the straight line which passes through the 'points' (y',x') and (y'',x'') in a plane-coordinate.)
- (3g) Drop all literals involving any function variable.

Let ϕ_3 be the formula obtained by execution of step 3. Then ϕ_3 is a formula involving only real variables, the arithmetic operators $+$, $-$, $\cdot$ and the predicates $=$, $<$, $\leq$, i.e. is a function of the polynomial

The first of these is the fact that the
  

theory of reals, which is decidable by Tarski's famous result [Tar51]. Thus to prove that the theory RMCF is decidable we only need to show that ϕ_2 and ϕ_3 are equisatisfiable, which we do as follows.

First of all, since all formulae introduced by step 3 are consequences of ϕ_2 , any model of ϕ_2 is also a model of ϕ_3 .

Conversely let M be a model of ϕ_3 . We will show how to extend M in a manner allowing all function variables to be interpreted in such a way as to satisfy all clauses of type $(f, f+, f>0, u-d, \text{strict}, \text{con})$. Also, for this, let V be the set of all real variables occurring in ϕ_2 as arguments of some function variable, and put $MV = \{My | y \in V\}$. For any function variable f occurring in ϕ_2 we define a corresponding function F having domain MV , by putting

(1) $F(My) = Mx$ if a literal $x = f(y)$ occurs in ϕ_2 .

The following easy lemma gives us what we now need.

Lemma 3.3. For any function variable f in ϕ_2 , the corresponding function F defined by (1) is single valued and has domain MV .

Proof: The only reason for which F could fail to be single valued is that ϕ_2 contains two literals $x = f(y)$, $x' = f(y')$, while $My = My'$ and $Mx \neq Mx'$. But this is impossible, since in such a case step (3c) would have introduced a literal $y = y' \rightarrow x = x'$, which force $Mx = Mx'$. Moreover, as observed just after the text of step 2, all function variables are evaluated in the same set of real variables, therefore it follows that the domain of any function F is just MV . $\square$

It just remains to extend our model functions F to the whole real axis. This can be done by the obvious procedure of linear

interpolation. More specifically let $\eta_1, \eta_2, \dots, \eta_r$ be the elements of MV, taken in increasing order. Then for any real η we put

$$(2) \quad (Mf)(\eta) = \begin{array}{ll} (F(\eta_1)) & \text{if } \eta < \eta_1 \\ \frac{\eta - \eta_i}{\eta_{i+1} - \eta_i} (F(\eta_{i+1}) - F(\eta_i)) + F(\eta_i) & \text{if } \eta_i \leq \eta < \eta_{i+1}, i \in \{1, \dots, r-1\} \\ F(\eta_r) & \text{if } \eta_r \leq \eta \end{array}$$

Then for any function variable f in ϕ_2 , Mf is a continuous function. Moreover, it follows at once by (1) and (2) that all clauses of type (f) in ϕ_2 are correctly modeled. We show now that clauses of type $(f+, f>, u-d, \text{strict}, \text{con})$ are also satisfied.

For this, let $f = g + h$ be a literal of ϕ_2 . We want to show that for any real η , $(Mf)(\eta) = (Mg)(\eta) + (Mh)(\eta)$. But this is clear if either $\eta < \eta_1$, or $\eta > \eta_r$, or $\eta = \eta_i$ for some $i = 1, \dots, r$, by (1), (2) and step (3a). Moreover, if $\eta_i < \eta < \eta_{i+1}$, for some $i = 1, \dots, r$, then

$$\begin{aligned} (Mf)(\eta) &= \frac{\eta - \eta_i}{\eta_{i+1} - \eta_i} (F(\eta_{i+1}) - F(\eta_i)) + F(\eta_i) \\ &= \frac{\eta - \eta_i}{\eta_{i+1} - \eta_i} (G(\eta_{i+1}) + H(\eta_{i+1}) - G(\eta_i) - H(\eta_i)) + G(\eta_i) + H(\eta_i) \\ &= \frac{\eta - \eta_i}{\eta_{i+1} - \eta_i} (G(\eta_{i+1}) - G(\eta_i)) + G(\eta_i) + \frac{\eta - \eta_i}{\eta_{i+1} - \eta_i} (H(\eta_{i+1}) - H(\eta_i)) + H(\eta_i) \\ &= (Mg)(\eta) + (Mh)(\eta) . \end{aligned}$$

This establishes that every clause $f = g + h$ is satisfied. It is also clear by (1), (2) and step (3b) that any literal $f > 0$ is correctly modeled. By (1), (2) and in view of the clauses added in steps (3d) and (3e) it follows easily that literals $(u-d, \text{strict})$ are also satisfied. Finally, let $\text{convex}(f, x, y)$ be a clause of ϕ_2 . If $Mx \geq My$, then M satisfies the clause $\text{convex}(f, x, y)$ which is void. Otherwise $Mx < My$; let $Mx = \eta_i$ and $My = \eta_j$, with $i < j$, $i, j \in \{1, \dots, r\}$. If Mf

 

coincides just with the straight line passing through the points $(\eta_i, (Mf)(\eta_i))$ and $(\eta_j, (Mf)(\eta_j))$, then clearly Mf is convex (strict convexity is not required!) in the interval $[\eta_i, \eta_j]$. On the other hand if Mf is not a straight line in the interval $[\eta_i, \eta_j]$, then by step (3f) the graph of Mf is the union of convex pieces. It follows therefore by an elementary argument that Mf is convex in the interval $[\eta_i, \eta_j]$. Thus in both cases Mf is convex in the interval $[Mx, My]$, which shows that M satisfies the clause $\text{convex}(f, x, y)$. Since M satisfies all purely arithmetic clauses in ϕ_2 of type $(+, \cdot, >)$, it follows at once that M is a model of ϕ_2 . Q.E.D.

This completes the proof of

Theorem 3.4. The theory RMCF is decidable.

Remark. Note that the restriction of the theory RMCF in which neither multiplication between real variables, nor the predicates 'convex' and 'concave' are allowed can be decided more efficiently by application of a linear programming algorithm ([Dan63]). The reason why a 'convex' and 'concave' predicate are not allowed in this simple case is that using these predicates and 'strict up' also it is possible to express the assertion that a function is a strictly increasing straight line, and hence it is an easy matter to express the product relationship among real variables. Hence a theory allowing convexity to be asserted along with the other predicates we have listed can be used to express any relationship admitted in Tarski arithmetic, making it plain that no such theory can admit a decision algorithm as efficient as simple linear programming.

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4. 'Extended Tarski' theory of reals with multivariate continuous functions.

For the sake of simplicity, we will only consider the case in which all function variables have the same arity n . The general case, that is when functions having different arity are allowed, can be easily reduced to this simpler theory by an obvious treatment by cases. Again we allow two kinds of variable, namely real variables $x, y, \dots$, and n -ary real function variables $f, g, \dots$. Constants $0, 1$, both numerical and functional, are also allowed. Numerical and functional terms are defined as in Definition 3.1, with the obvious replacement of condition (iii) by

(iii₁) If $t_1, \dots, t_n$ are numerical terms and f is a function variable, then $f(t_1, \dots, t_n)$ is a numerical term.

We define the family of formulae of the theory of reals with monotone multivariate continuous functions (RMMCF) as follows. An atom is an expression of one of the following forms: $t_1 = t_2$, $t_1 < t_2$, $\eta_1 = \eta_2$, $\eta_1 < \eta_2$, $\text{up}(\eta_1; t_1, u_1; \dots; t_n, u_n)$, $\text{strict_up}(\eta_1; t_1, u_1; \dots; t_n, u_n)$, where $t_1, t_2, \dots, t_n$, $u_1, \dots, u_n$ are numerical terms, and η_1, η_2 are functional terms. (Observe that no predicate 'convex' or 'concave' is allowed in this theory.) A formula in RMMCF is defined to be any Boolean combination of atoms. The intended meaning of the predicates up , strict_up , etc. which appear above is defined as follows. Let ϕ be a formula in RMMCF.

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Definition 4.1. An interpretation M of the symbols of ϕ is a model of ϕ if:

- (1) For any real variable x , the value Mx associated to x is a real number.
- (2) For any function variable f , Mf is an everywhere defined n -ary continuous real function.
- (3) The truth value assigned to ϕ by the following rule is the value true:

Let $t_1, t_2, \dots, t_n$, $u_1, \dots, u_n$ be numerical terms and η_1, η_2 be functional terms, and assume that $Mt_1, \dots, Mt_n$, $Mu_1, \dots, Mu_n$, $M\eta_1$, $M\eta_2$ are the interpretations assigned to them by the model M in the standard way. Then

- (3a) $t_1 = t_2$ (resp. $t_1 < t_2$) is true if and only if $Mt_1 = Mt_2$ (resp. $Mt_1 < Mt_2$).
- (3b) $\eta_1 = \eta_2$ (resp. $\eta_1 < \eta_2$) is true if and only if $M\eta_1 = M\eta_2$ (resp. $M\eta_1 < M\eta_2$) pointwise.
- (3c) $\text{up}(\eta_1; t_1, u_1; \dots; t_n, u_n)$ (resp. $\text{strict_up}(\eta_1; t_1, u_1; \dots; t_n, u_n)$) is true if and only if for every $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n) \in [Mt_1, Mu_1] \times \dots \times [Mt_n, Mu_n]$, with $\alpha_i \leq \beta_i$ for all $i = 1, \dots, n$, and $\alpha_j < \beta_j$ for some $j \in \{1, \dots, n\}$, $(M\eta_1)(\alpha_1, \dots, \alpha_n) \leq (M\eta_1)(\beta_1, \dots, \beta_n)$ (resp. $(M\eta_1)(\alpha_1, \dots, \alpha_n) < (M\eta_1)(\beta_1, \dots, \beta_n)$).

By applying a normalization process of the kind described in the preceding section, we can reduce the satisfiability problem for the theory RMMCF to the satisfiability problem for conjunctions of literals of the following types:

- (+) $x = y + z$

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($\cdot$) $x = y \cdot z$

($>$) $x > 0$

(f) $x = f(y_1, \dots, y_n)$

($f+$) $f = g + h$

($f \neq$) $f \neq g$

($f >$) $f > 0$

(up) $\text{up}(f; x_1, y_1; \dots; x_n, y_n)$

(strict_up) $\text{strict_up}(f; x_1, y_1; \dots; x_n, y_n)$

($\neg \text{up}$) $\neg \text{up}(f; x_1, y_1; \dots; x_n, y_n)$

($\neg \text{strict_up}$) $\neg \text{strict_up}(f; x_1, y_1; \dots; x_n, y_n)$

where $x, y, z, x_1, \dots, x_n, y_1, \dots, y_n$ are real variables, and f, g, h are function variables.

In analogy with the theory of reals with monotone and convex continuous functions discussed in the preceding section, we will prove that the theory RMMCF is decidable by showing how a conjunction of literals of the types $(+)$ - $(\neg \text{strict_up})$ listed above can be reduced to a conjunction involving only literals $(+, \cdot, <)$. Tarski's result concerning the real field (see [Tar51]) will then imply the decidability of RMMCF .

Let ϕ be a conjunction of literals each of which is of one of the types $(+)$ - $(\neg \text{strict_up})$ above. We apply the following simplification steps to ϕ , to eliminate all literals other than those of type $(+, \cdot, <)$.

The first two are the same as in the previous section.
 The third is the same as in the previous section.
 The fourth is the same as in the previous section.
 The fifth is the same as in the previous section.
 The sixth is the same as in the previous section.
 The seventh is the same as in the previous section.
 The eighth is the same as in the previous section.
 The ninth is the same as in the previous section.
 The tenth is the same as in the previous section.

Simplification Step 1.

(1a) For any literal $f \neq g$ occurring in ϕ , introduce new variables $x_1, x_2, y_1, \dots, y_n$ and replace $f \neq g$ by the formula $x_1 = f(y_1, \dots, y_n) \ \& \ x_2 = g(y_1, \dots, y_n) \ \& \ x_1 \neq x_2$.

(1b) Replace any literal $\neg \uparrow \text{up}(f; x_1, y_1; \dots; x_n, y_n)$ (resp. $\neg \uparrow \text{strict_up}(f; x_1, y_1; \dots; x_n, y_n)$) by the formula

$$A \ \& \ u_1 > u_2 \quad (\text{resp. } A \ \& \ u_1 \geq u_2) ,$$

$$\text{where } A = \bigwedge_{i=1}^n (x_i \leq w_i \leq z_i \leq y_i) \ \& \ \bigvee_{i=1}^n (w_i < z_i)$$

$$\ \& \ u_1 = f(w_1, \dots, w_n) \ \& \ u_2 = f(z_1, \dots, z_n)$$

and where $w_1, \dots, w_n, z_1, \dots, z_n, u_1, u_2$ are newly introduced variables. (Note that the formulae above simply express the fact that the interval $[x_1, y_1] \times \dots \times [x_n, y_n]$ contains two points $(w_1, \dots, w_n), (z_1, \dots, z_n)$ which falsify the predicate $\text{up}(f; x_1, y_1; \dots; x_n, y_n)$ (resp. $\text{strict_up}(f; x_1, y_1; \dots; x_n, y_n)$) according to (3c) of Definition 4.1.)

Let ϕ_1 be the formula into which ϕ is transformed by application of this simplification step. It is clear that ϕ and ϕ_1 are equisatisfiable. It therefore follows that ϕ_1 can be taken to be a conjunction of literals of the following types (+, ., >, f, f+, f>, up, strict_up). Simplification Step 1 eliminates literals of type (f≠, $\neg \uparrow \text{up}$, $\neg \uparrow \text{strict_up}$). To eliminate the remaining clauses involving function variables, namely those of type (f, f+, f>, up, strict_up), we then subject ϕ_1 to the following

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Simplification Step 2.

For every $i = 1, \dots, n$, let V_i be the set of real variables which occur either as the i -th argument of some function variable in ϕ_1 or as left or right $(i+1)$ -st argument of a predicate of type `up`, `strict_up` in ϕ_1 . In other words, V_i is the set of all variables which take values in the i -th factor space of the product space $\mathbb{R}^n$. For any $\xi = (\xi_1, \dots, \xi_n) \in V_1 \times \dots \times V_n$ and any function variable f occurring in ϕ_1 , introduce a new real variable $z_{f,\xi}$, and add the formula $z_{f,\xi} = f(\xi_1, \dots, \xi_n)$, unless a literal $y = f(\xi_1, \dots, \xi_n)$ exists already.

Call the resulting formula ϕ_2 . Clearly ϕ_2 is equisatisfiable with ϕ_1 . Once this has been done, apply the following

Simplification Step 3.

- (3a) For each quadruple of literals $x = f(y_1, \dots, y_n)$, $x' = g(y_1, \dots, y_n)$, $x'' = h(y_1, \dots, y_n)$, $f = g + h$, add $x = x' + x''$.
- (3b) For each pair of literals $x = f(y_1, \dots, y_n)$, $f > 0$, add $x > 0$.
- (3c) For each pair of literals $x = f(y_1, \dots, y_n)$, $x' = f(y'_1, \dots, y'_n)$ add the formula

$$\bigwedge_{i=1}^n (y_i = y'_i) \rightarrow x = x'.$$

- (3d) For each triple of literals $x = f(y_1, \dots, y_n)$, $x' = f(y'_1, \dots, y'_n)$, `up`(f ; v_1, w_1 ; ...; v_n, w_n) (resp. `strict_up`(f ; v_1, w_1 ; ...; v_n, w_n)) add the formula

$$B \rightarrow x \leq x' \quad (\text{resp. } B \rightarrow x < x')$$

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$$\text{where } B \equiv \bigwedge_{i=1}^n (v_i \leq y_i \leq y_i^! \leq w_i) \ \& \ \bigvee_{i=1}^n (y_i < y_i^!) .$$

(3e) From ϕ_2 , drop all literals involving any function variable.

Let ϕ_3 be the formula produced by Simplification Step 3. Observe that ϕ_3 is a formula of the polynomial theory of reals, which is known to be decidable [Tar51]. Therefore in order to prove the decidability of the theory RMMCF, it suffices to show that Simplification Step 3 preserves satisfiability, namely that ϕ_2 and ϕ_3 are equisatisfiable. This can be done as follows. Since all formulae added by step 3 are consequences of ϕ_2 (cfr. Definition 4.1), it follows at once that any model of ϕ_2 can be extended to a model of ϕ_3 . Conversely, let M be a model of ϕ_3 .

It follows in virtue of the formulae added in simplification steps 2 and 3 that the presence of a formula $f = g + h$ in ϕ implies that

$$(Mf)(u_1, \dots, u_n) = (Mg)(u_1, \dots, u_n) + (Mh)(u_1, \dots, u_n)$$

for all $(u_1, \dots, u_n) \in D$, where $D = MV_1 \times \dots \times MV_n$. Similarly, the presence of a formula $f > 0$ in ϕ implies that $(Mf)(u_1, \dots, u_n) > 0$ for all $(u_1, \dots, u_n) \in D$; and if $\text{up}(f; x_1, y_1; \dots; x_n, y_n)$ (resp. $\text{strict_up}(f; x_1, y_1; \dots; x_n, y_n)$) belongs to ϕ , then $(Mf)(u_1, \dots, u_n) \geq (Mf)(v_1, \dots, v_n)$ (resp. $(Mf)(u_1, \dots, u_n) > (Mf)(v_1, \dots, v_n)$) for any two distinct $(u_1, \dots, u_n) \in D$, $(v_1, \dots, v_n) \in D$ such that $My_i \geq u_i \geq v_i \geq Mx_i$ for all $i = 1, \dots, n$.

The fact that ϕ_2 and ϕ_3 are equisatisfiable will therefore follow immediately from the following elementary extension lemma.

Lemma 4.2. For each $i = 1, \dots, n$ let $u_1^{(i)} < \dots < u_{m_i}^{(i)}$ be a finite, strictly increasing sequence of real numbers. Let D be the product set

 

$$D = \{u_1^{(1)}, \dots, u_{m_1}^{(1)}\} \times \dots \times \{u_1^{(n)}, \dots, u_{m_n}^{(n)}\}.$$

Let Φ be the space of all real valued functions defined on D . Then there exists a systematic procedure for extending each of the functions $f \in \Phi$ to a continuous real-valued function Ef defined on all of R^n , which is such that $E(f+g) = Ef + Eg$, and $(Ef)(x_1, \dots, x_n) \geq (Eg)(x_1, \dots, x_n)$ (resp. $(Ef)(x_1, \dots, x_n) > (Eg)(x_1, \dots, x_n)$) for all $(x_1, \dots, x_n) \in R^n$ whenever $f(u_1, \dots, u_n) \geq g(u_1, \dots, u_n)$ (resp. $f(u_1, \dots, u_n) > g(u_1, \dots, u_n)$) for all $(u_1, \dots, u_n) \in D$. Moreover, if $(X_1, \dots, X_n) \in D$, $(Y_1, \dots, Y_n) \in D$ and f satisfies $f(u_1, \dots, u_n) \geq f(v_1, \dots, v_n)$ (resp. $f(u_1, \dots, u_n) > f(v_1, \dots, v_n)$) for any two distinct $(u_1, \dots, u_n) \in D$, $(v_1, \dots, v_n) \in D$ such that $Y_i \geq u_i \geq v_i \geq X_i$ for all $i = 1, \dots, n$, then Ef satisfies $(Ef)(y_1, \dots, y_n) \geq (Ef)(x_1, \dots, x_n)$ (resp. $(Ef)(y_1, \dots, y_n) > (Ef)(x_1, \dots, x_n)$) for any two distinct $(y_1, \dots, y_n) \in R^n$, $(x_1, \dots, x_n) \in R^n$ such that $Y_i \geq y_i \geq x_i \geq X_i$, $i = 1, \dots, n$.

Proof: We proceed by induction on the dimension n . Suppose that $n = 1$. Then $E_1 f$ can be defined as the function which is linear in every interval between two points $u_1^{(1)}$, $u_{1+1}^{(1)}$ and which agrees with f at each of the points $u_1^{(1)}$. It is also appropriate to define Ef to be constant in the two semi-infinite intervals $x \leq u_1^{(1)}$ and $x \geq u_{m_1}^{(1)}$.

This same definition can be applied to the space of functions f with values in any partially ordered vector space, and, in particular, to functions f whose values belong to the space of continuous real-valued functions of $n-1$ variables $g(u_2, \dots, u_n)$. In all such cases we have $E_1(f+g) = E_1 f + E_1 g$ and $E_1 f \geq E_1 g$ (resp. $E_1 f > E_1 g$) at every point of the real axis if $f \geq g$ (resp. $f > g$) at every point of its

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finite domain. Let T be the transformation which maps each function $f(u_1, \dots, u_n)$ defined on D into the vector-valued function $(Tf)(u_1)$ defined by

$$(Tf)(u_1)(u_2, \dots, u_n) = f(u_1, \dots, u_n) ;$$

here it should be understood that the values $(Tf)(u_1)$ lie in the space E_{n-1} of real-valued functions on the domain

$$D_2 = \{u_1^{(2)}, \dots, u_{m_2}^{(2)}\} \times \dots \times \{u_1^{(n)}, \dots, u_{m_n}^{(n)}\} .$$

Let E_{n-1} be the inductively defined extension operator which transforms functions in E_{n-1} into real-valued functions defined on all of R^{n-1} , and put

$$(*) \quad (E_n f) = E_1(E_{n-1}(Tf(\cdot))) ,$$

where $E_{n-1}(Tf(\cdot))$ simply denotes the function whose value at u_1 is $E_{n-1}(Tf(u_1))$. Then it plainly follows by induction that $E_n(f+g) = E_n f + E_n g$, and $(E_n f)(x_1, \dots, x_n) \geq (E_n g)(x_1, \dots, x_n)$ (resp. $(E_n f)(x_1, \dots, x_n) > (E_n g)(x_1, \dots, x_n)$) for all $(x_1, \dots, x_n) \in R^n$ if $f(u_1, \dots, u_n) \geq g(u_1, \dots, u_n)$ (resp. $f(u_1, \dots, u_n) > g(u_1, \dots, u_n)$) for all $(u_1, \dots, u_n) \in D$.

Next suppose that $f(u_1, \dots, u_n) \geq f(v_1, \dots, v_n)$ (resp. $f(u_1, \dots, u_n) > f(v_1, \dots, v_n)$) for any two distinct $(u_1, \dots, u_n) \in D$, $(v_1, \dots, v_n) \in D$ such that $Y_i \geq u_i \geq v_i \geq X_i$ for all $i = 1, \dots, n$. They by induction

$$(E_{n-1} Tf(u_1))(y_2, \dots, y_n) \geq (E_{n-1} Tf)(u_1)(x_2, \dots, x_n) \quad (\text{resp.})$$

 

$(E_{n-1} \text{Tr}(u_1))(y_2, \dots, y_n) > (E_{n-1} \text{Tr}(u_1))(x_2, \dots, x_n)$ for each $u_1 \in \{u_1^{(1)}, \dots, u_{m_1}^{(1)}\}$ and any two distinct $(x_2, \dots, x_n) \in R^{n-1}$, $(y_2, \dots, y_n) \in R^{n-1}$ such that $Y_1 \geq u_1 \geq X_1$, and $Y_i \geq y_i \geq x_i \geq X_i$ for all $i = 2, \dots, n$. It follows at once from the definition of E_1 and from (*) that $(E_n f)(x_1, y_2, \dots, y_n) \geq (E_n f)(x_1, x_2, \dots, x_n)$ (resp. $(E_n f)(x_1, y_2, \dots, y_n) > (E_n f)(x_1, x_2, \dots, x_n)$) for any two distinct $(x_2, \dots, x_n) \in R^{n-1}$, $(y_2, \dots, y_n) \in R^{n-1}$ and any x_1 such that $Y_1 \geq x_1 \geq X_1$ and $Y_i \geq y_i \geq x_i \geq X_i$ for $i = 2, \dots, n$. A similar argument shows that $(E_n f)(y_1, \dots, y_n) \geq (E_n f)(x_1, \dots, x_n)$ whenever $Y_i \geq y_i \geq x_i \geq X_i$ for all $i = 1, \dots, n$, and that $(E_n f)(y_1, \dots, y_n) > (E_n f)(x_1, \dots, x_n)$ whenever $Y_i \geq y_i \geq x_i \geq X_i$ for all $i = 1, \dots, n$ and $(y_1, \dots, y_n), (x_1, \dots, x_n)$ are distinct, provided that $f(u_1, \dots, u_n) > f(v_1, \dots, v_n)$ for any two distinct $(u_1, \dots, u_n) \in D$, $(v_1, \dots, v_n) \in D$ such that $Y_i \geq u_i \geq v_i \geq X_i$ for all $i = 1, \dots, n$. This completes the proof of Lemma 4.2, and with it the decidability of the theory RMMCF. Q.E.D.

Summarizing we have

Theorem 4.3. The theory RMMCF is decidable.

Remark. RMMCF remains decidable even in the presence of slightly more general kind of up, strict_up predicates. More precisely consider a predicate $\text{monotone}(f; a_1: x_1, y_1; \dots; a_n: x_n, y_n)$ involving a function variable f , real variables $x_1, \dots, x_n, y_1, \dots, y_n$, and constants $a_1, \dots, a_n$ which can take the values 'up', 'strict_up', 'down', 'strict_down'. By definition $\text{monotone}(f; a_1: x_1, y_1; \dots; a_n: x_n, y_n)$ is satisfied if for every $i \in \{1, \dots, n\}$ and every $\xi, \eta \in [x_i, y_i] \times \dots \times [x_n, y_n]$ such that $\xi_i < \eta_i$ and $\xi_j = \eta_j$ if $j \neq i$, the condition

$$a_i = \left\{ \begin{array}{l} \text{up} \\ \text{strict_up} \\ \\ \text{down} \\ \text{strict_down} \end{array} \right\} \text{implies } (Mf)(\xi) \left\{ \begin{array}{l} \leq \\ < \\ \\ \geq \\ > \end{array} \right\} (Mf)(\eta)$$

Much the same construction as above allows us to prove that the theory RMMCF extended to allow this more general predicate 'monotone' remains decidable.

5. Totally and partially ordered sets with monotone functions.

In this section we take up a more abstract problem suggested by the preceding considerations. More specifically, we consider two elementary theories of partially and totally ordered sets, beginning with the 'theory of totally ordered sets with monotone functions'. This theory allows individual variables $x, y, \dots$ which take elements of a totally ordered set S as values, and also variables denoting (monotone) functions $f_1, f_2, \dots$ which are modeled by functions defined on S and assuming values in S . More formally

Definition 5.1. Terms of the 'theory of totally ordered sets with monotone functions' (TOSMF) are:

- (i) individual variables;
- (ii) expressions of the form $f(t)$, where f is a function variable and t is a term.

Definition 5.2. The atomic formulas of TOSMF are obtained as follows:

- (i) if t_1 and t_2 are terms, then $t_1 = t_2$, $t_1 < t_2$ are atomic formulas;
- (ii) if f is a function variable, then $\text{up}(f)$, $\text{strict_up}(f)$, $\text{down}(f)$, $\text{strict_down}(f)$ are atomic formulas.

The theory TOSMF is then the Boolean closure of the set of atomic formulas defined above.

Let ϕ be a formula of the theory TOSMF.

Definition 5.3. An interpretation M with domain D is said to satisfy ϕ if:

- (i) D is a totally ordered set;

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- (ii) for every individual variable x in ϕ , $Mx \in D$;
- (iii) for every function variable f occurring in ϕ , Mf is a function from D to D ;
- (iv) the formula $M\phi$ obtained from ϕ by substituting every individual variable x by Mx and function variable f by Mf is true. Here it should be understood that

(iv₁) $\text{up}(f)$ (resp. $\text{strict_up}(f)$) is true in the interpretation M if and only if Mf is a nondecreasing (resp. strictly increasing) function.

(iv₂) $\text{down}(f)$ (resp. $\text{strict_down}(f)$) is true in the interpretation M if and only if Mf is a nonincreasing (resp. strictly decreasing) function.

Using a normalization process like that described in Section 2, it is easy to show that the satisfiability problem for the theory TOSMF can be reduced to the satisfiability problem for sets of conjunctions of literals each of which has one of the following forms:

- ($<$) $x < y$
- ($=$) $x = y$
- (f) $x = f(y)$
- ($u-d$) $\text{up}(f), \text{down}(f)$
- (strict) $\text{strict_up}(f), \text{strict_down}(f)$
- ($\neg u-d$) $\neg \text{up}(f), \neg \text{down}(f)$
- ($\neg \text{strict}$) $\neg \text{strict_up}(f), \neg \text{strict_down}(f)$.

Let ϕ be a conjunction of literals of the types ($<$, $=$, f , $u-d$, strict , $\neg u-d$, $\neg \text{strict}$). As usual we can easily eliminate clauses in ϕ of types ($\neg u-d$, $\neg \text{strict}$), and thus can assume that no clause of the type ($\neg u-d$, $\neg \text{strict}$) occurs in ϕ . We will show that ϕ is satisfiable

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(according to the satisfiability definition given above) if and only if ϕ has a real model, that is a model whose domain is the real axis $\mathbb{R}$ with the standard order in it. Plainly any real model is a model of ϕ . Conversely, assume that M is a model of ϕ . Then a real model of ϕ can be built as follows. Let $T = \{Mx \mid x \text{ occurs in } \phi\} \cup \{(Mf)(Mx) \mid f, x \text{ occur in } \phi\}$. Clearly T is a totally ordered finite subset of the domain D of M . Hence T can be embedded in $\mathbb{R}$. Let $I: T \rightarrow \mathbb{R}$ be such an embedding. For any individual variable x occurring in ϕ , put

$$M^*x = I(Mx)$$

Moreover with any function variable f occurring in ϕ , we associate a function $F: \{M^*x \mid x \text{ occur in } \phi\} \rightarrow \mathbb{R}$ defined by

$$F(M^*x) = I((Mf)(Mx)).$$

We begin by showing that if $\text{up}(f)$ is a conjunct of ϕ , then F is monotone nondecreasing on $\{M^*x \mid x \text{ occurs in } \phi\}$. Indeed, let $M^*x_1 \leq M^*x_2$ for some pair of variables x_1, x_2 occurring in ϕ . Then $Mx_1 = I^{-1}(M^*x_1) \leq I^{-1}(M^*x_2) = Mx_2$, so that $F(M^*x_1) = I((Mf)(Mx_1)) \leq I((Mf)(Mx_2)) = F(M^*x_2)$. Similarly it follows that if $\text{strict_up}(f)$, $\text{down}(f)$, or $\text{strict_down}(f)$ appear in ϕ , then F is monotone increasing, nonincreasing or decreasing (respectively). Therefore F can be extended to an everywhere continuous function $\bar{F}$ in such a way that if $\text{up}(f)$, $\text{strict_up}(f)$, $\text{down}(f)$, $\text{strict_down}(f)$ occur in ϕ , then $\bar{F}$ is respectively monotone nondecreasing, increasing, nonincreasing or decreasing. Finally we put $M^*f = \bar{F}$, for any function variable f occurring in ϕ . It then is an easy matter to verify that M^* is a real model of ϕ . Hence the decidability of the theory TOSMF follows from Theorem 3.4.

Therefore we have

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Theorem 5.4. The theory TOSMF is decidable.

Next we consider the somewhat different theory of partially ordered sets with (monotone) functions (POSMF) and show that its satisfiability problem is decidable also. More precisely we will refer to those partially ordered sets in which any two elements have both a lower and an upper bound. The syntax of the theory POSMF is the same as the one of the theory TOSMF, with the only exception that no 'strict' predicate is allowed in this case. Moreover we define satisfiability as follows.

Definition 5.5. Let ϕ be a formula of the theory POSMF. An interpretation M with domain D is said to satisfy ϕ if:

- (i) D is a partially ordered set such that any two elements in D have both a lower and an upper bound, and in which the relationship $x \leq y \leq x$ implies that $x = y$.
- (ii) for every individual variable x in ϕ , $Mx \in D$;
- (iii) for every function variable f occurring in ϕ , Mf is a function from D to D ;
- (iv) the formula $M\phi$ obtained from ϕ by replacing x by Mx and f by Mf , wherever an individual variable x or a function variable f occurs in ϕ , is true, and where we agree that $\text{up}(f)$ (resp. $\text{down}(f)$) is true in the interpretation M if and only if Mf is a monotone nondecreasing (resp. nonincreasing) function;

Normalizing as before the satisfiability problem for the theory POSMF is easily reduced to the satisfiability problem for sets of literals of the form

- ($\leq$) $x \leq y$
- ($\not\leq$) $x \not\leq y$ (i.e. $\neg(x \leq y)$)
- ($=$) $x = y$
- ($\neq$) $x \neq y$
- (f) $x = f(y)$
- (u-d) $\text{up}(f), \text{down}(f)$
- ($\neg$ u-d) $\neg \text{up}(f), \neg \text{down}(f)$.

Moreover, since clauses ($\neg$ u-d) can be eliminated quite easily (cf. the simplification step 1 of the preceding section) we can further restrict our attention to conjunctions of literals of the type ($\leq$) - (u-d). Hence let ϕ be such a conjunction. Let ϕ_1 be the formula obtained from ϕ after execution of the following

Simplification Step.

- (a) For every pair of clauses of ϕ of the form $x = f(y), x' = f(y')$ add the formula $y = y' \rightarrow x = x'$
- (b) For every triple of clauses of ϕ of the form $\text{up}(f)$ (resp. $\text{down}(f)$), $x = f(y), x' = f(y')$ add the formula $y \leq y' \rightarrow x \leq x'$ (resp. $y \leq y' \rightarrow x' \leq x$).
- (c) For every quadruple of clauses of ϕ of the form $\text{up}(f), \text{down}(f), x = f(y), x' = f(y')$ add the formula $x = x'$.
- (d) Drop all clauses of ϕ involving function variables.

Clearly any model of ϕ is also a model of ϕ_1 , since formulas added by the simplification substeps (a), (b) and (c) are consequences of ϕ . For example, by (i) of Definition 5.5 it follows that any function which is both increasing and decreasing must be a constant function,

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[illegible]

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which shows the correctness of substep (c). Let $\bar{\phi}_1$ be the formula obtained from ϕ_1 by replacing every occurrence of the relation symbols $\leq$, $\not\leq$ with $\subseteq$ and $\not\subseteq$ respectively, where $\subseteq$ is the ordinary set inclusion symbol.

We will now show that if ϕ_1 has a partially ordered model, then $\bar{\phi}_1$ has a set model defined as follows:

Definition 5.6. An interpretation $\bar{M}$ with domain D is a set model of $\bar{\phi}_1$ if

- (i) there is a set S such that $D = \text{pow}(S)$ (where $\text{pow}(S)$ is the set of all subsets of S);
- (ii) for every individual variable x in $\bar{\phi}_1$, $\bar{M}x \in D$;
- (iii) the formula $\bar{M} \bar{\phi}_1$ obtained from $\bar{\phi}_1$ by replacing x by $\bar{M}x$ for each individual variable x occurring in $\bar{\phi}_1$ is true.

To prove our assertion, let M be a partially ordered model of ϕ_1 having domain S . We will build a set model $\bar{M}$ of $\bar{\phi}_1$ with domain $D = \text{pow}(S)$. Let $h: S \rightarrow \text{pow}(S)$ be the mapping defined by $h(s) = \{t \in S \mid t \leq s\}$ (where $\leq$ is the order relation in S). For every individual variable x in $\bar{\phi}_1$, we put

$$\bar{M}x = h(Mx) .$$

Let $x \subseteq y$ be a clause of $\bar{\phi}_1$. Then $x \leq y$ is a clause in ϕ_1 , and since $Mx \leq My$, it follows by the transitivity of $\leq$ and the definition of $h: S \rightarrow \text{pow}(S)$ that $h(Mx) \subseteq h(My)$, i.e. $\bar{M}x \subseteq \bar{M}y$. Therefore $\bar{M}$ satisfies all literals in $\bar{\phi}_1$ of the type $(\subseteq)$. Similarly it can be shown that $\bar{M}$ also satisfies all literals in $\bar{\phi}_1$ of the remaining types including the

1. The first part of the paper
describes the general situation
of the country.
2. The second part of the paper
describes the situation of the
country in the year 1950.
3. The third part of the paper
describes the situation of the
country in the year 1960.

1. The first part of the paper
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3. The third part of the paper
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country in the year 1960.

implicative clauses introduced in step (a) and (b), which proves that $\bar{M}$ is a set model of $\bar{\phi}_1$.

Moreover any set model $\bar{M}$ of $\bar{\phi}_1$ can be used to define a partially ordered model M of ϕ . Indeed, the relation $\subseteq$ is a partial order of the domain $D = \text{pow}(S)$ of $\bar{M}$, so that all literals in ϕ of the form $(\leq, \not\leq, =, \neq)$ are correctly modeled by $\bar{M}$ if $\leq$ is interpreted as set inclusion, and clearly $x \cap y$ and $x \cup y$ are respectively a lower and an upper bound of $x, y \in D$. Hence all we need to do in order to get a model of ϕ is to extend $\bar{M}$ to function variables in such a way that clauses of types $(f, u-d)$ are correctly modeled too.

For every function variable f occurring in ϕ for which $\text{up}(f)$ occurs in ϕ while $\text{down}(f)$ does not occur in ϕ , and for any $s \in S$, put

$$(5.1) \quad (\bar{M}f)(s) = \bigcup_{x \in A_s} \bar{M}x$$

where $A_s = \{x \mid x = f(y) \text{ occurs in } \phi \text{ and } \bar{M}y \subseteq s\}$.

For every function variable f occurring in ϕ for which $\text{down}(f)$ occurs in ϕ while $\text{up}(f)$ does not occur in ϕ , and for any $s \in S$, put

$$(5.2) \quad (\bar{M}f)(s) = \bigcap_{x \in A_s} \bar{M}x$$

If both $\text{up}(f)$ and $\text{down}(f)$ occur in ϕ , put

$$(5.3) \quad (\bar{M}f)(s) = \bar{M}x,$$

for some $x = f(y)$ occurring in ϕ . Observe that (5.3) does not depend

on the clause $x = f(y)$ chosen since by the Simplification Substep (c) $\bar{M}x = \bar{M}x'$ if both $x = f(y)$ and $x' = f(y)$ occur in ϕ . If neither $up(f)$ nor $down(f)$ occurs in ϕ , define $\bar{M}f$ to be any extension to the whole of D of the set-to-set mapping F , defined by

$$F(\bar{M}y) = \bar{M}x \text{ if } x = f(y) \text{ occurs in } \phi.$$

(Note that by the Simplification Substep (a), this definition is legal.) Plainly if neither $up(f)$ nor $down(f)$ occurs in ϕ , all clauses of type (f) are satisfied by this definition.

We must show that if $up(f)$ is in ϕ but $down(f)$ does not occur in ϕ , then $\bar{M}f$ is monotone nondecreasing, with respect to the relation $\subseteq$. To this end, let $s, t \in S$, such that $s \subseteq t$. Clearly $A_s \subseteq A_t$, and therefore by (5.1)

$$(\bar{M}f)(s) = \bigcup_{x \in A_s} \bar{M}x \subseteq \bigcup_{x \in A_t} \bar{M}x = (\bar{M}f)(t),$$

which shows that $\bar{M}f$ is monotone nondecreasing. Next, suppose that $up(f)$ occurs in ϕ and that $x' = f(y')$ is a literal of ϕ . To be sure that all type (f) clauses are satisfied in this case also we must prove that $(\bar{M}f)(\bar{M}y') = \bar{M}x'$. Let $x = f(y)$ be any literal in ϕ such that $\bar{M}y \subseteq \bar{M}y'$. Since $\bar{\phi}_1$ contains the conjuncts $y \subseteq y' \rightarrow x \subseteq x'$ and $y = y' \rightarrow x = x'$ (which are introduced into $\bar{\phi}_1$ by the Simplification Substeps (b) or (a)), it follows that $\bar{M}x \subseteq \bar{M}x'$. This shows that $(\bar{M}f)(\bar{M}y') \subseteq \bar{M}x'$. However, $(\bar{M}f)(\bar{M}y') \supseteq \bar{M}x'$, since $x' = f(y')$ occurs in ϕ and $\bar{M}y' \subseteq \bar{M}y'$. Thus $(\bar{M}f)(\bar{M}y') = \bar{M}x'$.

A very similar argument shows that if $down(f)$ occurs in ϕ but $up(f)$ does not, then (5.2) correctly models both the statement $down(f)$ and all clauses $x = f(y)$ involving the function variable f . The case

in which both $\text{up}(f)$ and $\text{down}(f)$ occur in ϕ is even easier. Taken all together, this shows that $\bar{M}$ models the set ϕ of statements correctly.

It follows that ϕ is satisfiable if and only if $\bar{\phi}_1$ has a set model. Since $\bar{\phi}_1$ is a trivial subset of the decidable theory MLS (see [FOS 80], cf. also [Qui50]), we can conclude that

Theorem 5.7. The theory POSMF is decidable.

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